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On the origin of the mass gap for non-Abelian gauge theories in (2+1) dimensions

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Abstract

An analysis of how the mass gap could arise in pure Yang-Mills theories in two spatial dimensions is given.

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The study of non-Abelian gauge theories in two spatial dimensions, in particular the question of how a mass gap could arise in these theories, is interesting for at least two reasons: it is a useful guide to the more realistic case of three dimensions and secondly these theories can be an approximation to the high temperature phase of Chromodynamics with the mass gap serving as the magnetic mass. In a recent paper, we considered the Hamiltonian analysis of (2+1)-dimensional gauge theories in a gauge-invariant matrix parametrization of the fields [1]. The kinetic term of the Hamiltonian, which is the Laplacian on the space of gauge-invariant configurations, could be explicitly constructed in our parametrization. By considering eigenstates of the Laplacian, one could then see how a mass gap could arise in these theories. In this letter, we rederive the key results directly in terms of the gauge potentials and electric fields. A general expression for the Hamiltonian is also given in terms of a current and derivatives with respect to it.

We consider the Hamiltonian version of an $SU(N)$-gauge theory in the $A_0 = 0$ gauge. The gauge potential is written $A_i = -it^a A_i^a$, $i = 1, 2$, where t^a are hermitian $N \times N$-matrices which form a basis of the Lie algebra of $SU(N)$ with $[t^a, t^b] = if^{abc}t^c$, $\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$. The Hamiltonian can be written as

$$
\begin{aligned}
\mathcal{H} &= T \ + V \\
T &= \frac{e^2}{2} \int d^2x \ E_i^a E_i^a \\
V &= \frac{1}{2e^2} \int d^2x \ B^a B^a
\end{aligned}
\tag{1}
$$

where e is the coupling constant and $B^a = \frac{1}{2}\epsilon_{jk}(\partial_j A_k^a - \partial_k A_j^a + f^{abc} A_j^b A_k^c)$. We use complex coordinates $z = x_1 - ix_2$, $\bar{z} = x_1 + ix_2$ with the corresponding components of the potential, viz., $A_z = \frac{1}{2}(A_1 + iA_2)$, $A_{\bar{z}} = \frac{1}{2}(A_1 - iA_2) = -(A_z)^\dagger$. The wavefunctions for the physical states are gauge-invariant and have the inner product

$$
\langle 1|2\rangle \ = \ \int \ d\mu(\mathcal{C}) \ \Psi_1^* \Psi_2 \tag{2}
$$

Here $d\mu(\mathcal{C})$ is the volume measure on the configuration space $\mathcal{C}$ which is the space of gauge potentials $\mathcal{A}$ modulo the set of gauge transformations $\mathcal{G}_*$ which go to the identity at spatial infinity. The distance function on $\mathcal{A}$ is the standard Euclidean one

$$
ds^2 \ = \ \int d^2x \ \delta A_i^a \delta A_i^a \ = -8 \int \mathrm{Tr}(\delta A_z \delta A_{\bar{z}}) \tag{3}
$$

$d\mu(\mathcal{A})$ is the standard volume $[dA_z dA_{\bar{z}}]$ associated with (3) and $d\mu(\mathcal{C})$ should be obtained by dividing out the volume of $\mathcal{G}_*$, i.e., $d\mu(\mathcal{C}) = [dA_z dA_{\bar{z}}]/vol(\mathcal{G}_*)$.

The gauge potentials $A_z,\ A_{\bar{z}}$ can be parametrized in terms of complex $= SL(N,\mathbf{C})$-matrices $M,\ M^\dagger$ as

$$A_z = -\partial_z M M^{-1}, \qquad A_{\bar{z}} = M^{\dagger -1}\partial_{\bar{z}} M^\dagger \tag{4}$$

Given any gauge potential we can construct $M,\ M^\dagger$ at least as power series in the potential; for example,

$$\begin{aligned} M(x) &= 1 - \int G(x,z_1)A(z_1) + \int G(x,z_1)A(z_1)G(z_1,z_2)A(z_2) - \ldots \\ M^\dagger(x) &= 1 - \int \bar{A}(z_1)\bar{G}(z_1,x) + \int \bar{A}(z_1)\bar{G}(z_1,z_2)\bar{A}(z_2)\bar{G}(z_2,x) - \ldots \end{aligned} \tag{5}$$

where $A \equiv A_z,\ \bar{A} \equiv A_{\bar{z}}$ and $G, \bar{G}$ are Green's functions for $\partial_z,\ \partial_{\bar{z}}$ defined by

$$\begin{aligned} \bar{\partial}_x \bar{G}(x,y) \ &= \partial_x G(x,y) \ = \delta^{(2)}(x-y) \\ \bar{G}(x,x') = \frac{1}{\pi(z-z')}, &\qquad G(x,x') = \frac{1}{\pi(\bar{z}-\bar{z}')} \end{aligned} \tag{6}$$

Notice that the solutions (5) may be summed up and written as

$$\begin{aligned} M(x) \ &= \ 1 - \int_y D^{-1}(x,y)A(y) \\ M^\dagger(x) \ &= \ 1 - \int_y \bar{A}(y)\bar{D}^{-1}(y,x) \end{aligned} \tag{7}$$

where $D = \partial + A,\ \bar{D} = \bar{\partial} + \bar{A}$ are the covariant derivatives.

The matrix M (or $M^\dagger$) is not uniquely defined. M and $M\bar{V}(\bar{z})$, where $\bar{V}(\bar{z})$ is antiholomorphic, and likewise $M^\dagger$ and $V(z)M^\dagger$, lead to the same potential. Eventually we must ensure that this ambiguity of parametrization does not affect physical results.

Under a gauge transformation, $A \to A^g = gAg^{-1} - dg\ g^{-1}$ or equivalently, $M \to gM,\ M^\dagger \to M^\dagger g^{-1},\ g \in SU(N)$. $H = M^\dagger M$ is gauge-invariant and this will be the basic field variable of the theory.

We first consider the evaluation of $d\mu(\mathcal{C})$. In terms of $M, M^\dagger$, the metric (3) becomes

$$ds^2_{\mathcal{A}} \ = 8\int \mathrm{Tr}\left[D(\delta M M^{-1})\bar{D}(M^{\dagger -1}\delta M^\dagger)\right] \tag{8}$$

(Here $D, \bar{D}$ are in the adjoint representation.) The metric for $SL(N, \mathbf{C})$-matrices is given by

$$ds^2_{SL(N,\mathbf{C})} = 8 \int \mathrm{Tr}[(\delta M M^{-1})(M^{\dagger -1} \delta M^\dagger)] \tag{9}$$

The Haar measure $d\mu(M, M^\dagger)$ is the volume associated with this metric. The matrix H belongs to $SL(N, \mathbf{C})/SU(N)$. The metric on this space becomes

$$ds^2_H \; = 2 \int \; \mathrm{Tr}(H^{-1} \; \delta H)^2 \tag{10}$$

From (8-10) we see that

$$d\mu(\mathcal{C}) = \frac{d\mu(\mathcal{A})}{vol(\mathcal{G}_*)} = \frac{[dA_z dA_{\bar{z}}]}{vol(\mathcal{G}_*)} = (\det D_z D_{\bar{z}}) \frac{d\mu(M, M^\dagger)}{vol(\mathcal{G}_*)} \tag{11}$$

$d\mu(M, M^\dagger)/vol(\mathcal{G}_*)$ is given by the volume of the metric (10), viz. $d\mu(H) = \det r[\delta\varphi]$ where $H^{-1}\delta H = \delta\varphi^a r_{ab} t_b$, φ^a being real parameters for the hermitian matrices H. (A simple way to see this is the following. $\rho \equiv (M^{\dagger -1} dM^\dagger + dM M^{-1})$ is a differential form on $SL(N, \mathbf{C})$ which transforms as $\rho \rightarrow g\rho g^{-1}$ under $M \rightarrow gM$. Thus $\mathrm{Tr}(\rho^n) \; = \mathrm{Tr}(H^{-1} dH)^n$ are differential forms on $SL(N, \mathbf{C})/SU(N)$. The volume element is given by the differential form of maximal degree, i.e., for $n = (N^2 - 1)$. This is easily seen to be $\det r[d\varphi]$. For matrices which are functions of the spatial coordinates, as in our case, we have the product over the spatial points as well, giving the result stated.) With this result, we see from (11)

$$d\mu(\mathcal{C}) \;\; = \;\; d\mu(H) \;\; \det(D\bar{D}) \tag{12}$$

The problem is thus reduced to calculating $\det(D\bar{D})$. The answer to this is well known, $\det(D\bar{D}) = e^{2c_A \mathcal{S}(H)}$, $c_A \delta_{ab} = f_{amn} f_{bmn}$, upto an irrelevant constant [2]. $\mathcal{S}(H)$ is the Wess-Zumino-Witten (WZW) action for H.

$$\mathcal{S}(H) = \frac{1}{2\pi} \int \mathrm{Tr}(\partial H \bar{\partial} H^{-1}) + \frac{i}{12\pi} \int \epsilon^{\mu\nu\alpha} \mathrm{Tr}(H^{-1}\partial_\mu H H^{-1} \partial_\nu H H^{-1} \partial_\alpha H) \tag{13}$$

We thus have

$$d\mu(\mathcal{C}) = \; d\mu(H) \; e^{2c_A \mathcal{S}(H)} \tag{14}$$

This evaluation of $d\mu(\mathcal{C})$ has been given in reference [3].

Some of the details of the calculation of $\det(D\bar{D}) \equiv e^{\Gamma}$ are of interest in what follows. From the definition

$$\frac{\delta\Gamma}{\delta\bar{A}^a(x)} \;=\; (-i)\mathrm{Tr}\left[\bar{D}^{-1}(x,y)T^a\right]_{y\to x} \tag{15}$$

where $(T^a)_{mn} = -if_{amn}$. $\bar{D}^{-1}(x,y) = M^{\dagger-1}(x)M^{\dagger}(y)\bar{G}(x,y)$, so that $\mathrm{Tr}[\bar{D}^{-1}(x,y)T^a]_{y\to x}$ $= -(c_A/\pi)(M^{\dagger-1}\partial M^{\dagger})^a$. This leads to $\Gamma = 2c_A\mathcal{S}(M^{\dagger}) + f(M)$. This regularization is not gauge-invariant. For $d\mu(\mathcal{C})$, we need a gauge-invariant regularization of $\det(D\bar{D})$ such as covariant point-splitting or Pauli-Villars regulators.

$$\bar{D}^{-1}(x,y)_{Reg} = \left[\bar{D}^{-1}(x,y)\exp(A(x-y)+\bar{A}(\bar{x}-\bar{y}))\right]_{y\to x} \tag{16a}$$

$$= \left[\left(\bar{D}^{-1} + D(\mu^2 - D\bar{D})^{-1}\right)(x,y)\right]_{\mu^2\to\infty} \tag{16b}$$

Either one of these regulators gives

$$\mathrm{Tr}\left[\bar{D}^{-1}(x,y)T^a\right]_{y\to x} \;=\frac{1}{\pi}\mathrm{Tr}\left[(A - M^{\dagger-1}\partial M^{\dagger})T^a\right] \tag{17}$$

Correspondingly, we have $\frac{\delta\Gamma}{\delta A^a} = (-ic_A/\pi)2\mathrm{Tr}[t_a(A - M^{\dagger-1}\partial M^{\dagger})]$, leading to $\Gamma = 2c_A\mathcal{S}(H)$ and the result (14). We shall take (16a), viz. covariant point-splitting, as the regularization in what follows.

The inner product is now given by

$$\langle 1|2\rangle = \int d\mu(H)\; e^{2c_A\mathcal{S}(H)}\quad \Psi_1^*\Psi_2 \tag{18}$$

In an intuitive sense, at this stage, we can see how a mass gap could possibly arise. Writing $\Delta E,\ \Delta B$ for the root mean square fluctuations of the electric field E and the magnetic field B, we have, from the canonical commutation rules, $\Delta E\ \Delta B \sim k$, where k is the momentum variable. This gives an estimate for the energy

$$\mathcal{E} = \frac{1}{2}\left(\frac{e^2k^2}{\Delta B^2} + \frac{\Delta B^2}{e^2}\right) \tag{19}$$

For low lying states, we minimize $\mathcal{E}$ with respect to ΔB^2, $\Delta B^2_{min} \sim e^2k$, giving $\mathcal{E} \sim k$. This is, of course, the standard photon or perturbative gluon. However, for the non-Abelian theory, in calculating expectation values, we must take account of the factor

$e^{2c_A\mathcal{S}} \approx \exp[-(c_A/2\pi)\int B(1/k^2)B + ...]$. For low k, this factor controls fluctuations in B, giving $\Delta B^2 \sim k^2(\pi/c_A)$. In other words, eventhough $\mathcal{E}$ is minimized around $\Delta B^2 \sim k$, probability is concentrated around $\Delta B^2 \sim k^2(\pi/c_A)$. For the expectation value of the energy, we then find $\mathcal{E} \sim (e^2 c_A/2\pi) + \mathcal{O}(k^2)$. Thus the kinetic term in combination with the measure factor $e^{2c_A\mathcal{S}}$ could lead to a mass gap. (This argument is very similar to how a mass is obtained for longitudinal plasma oscillations.) The argument is not rigorous; many terms, such as the non-Abelian contributions to the commutators and $\mathcal{S}(H)$, have been neglected. Nevertheless, we expect this to capture the essence of how a mass gap could arise.

Matrix elements calculated with the inner product (18) are correlation functions of a WZW-model for hermitian matrices. They can be calculated by analytic continuation of the results for unitary matrices. For hermitian matrices, $e^{(k+2c_A)\mathcal{S}}$ is what corresponds to the action $e^{k\mathcal{S}(U)}$ for unitary matrices. The analogue of the renormalized level $\kappa = k + c_A$ of the unitary WZW model is $-(k + c_A) = -\kappa$. The correlators are obtained by the continuation $\kappa \to -\kappa$. In the present case, we have only $e^{2c_A\mathcal{S}}$, hence we must also take $k \to 0$. In the unitary case, correlators involving primary fields belonging to the nonintegrable representations of the current algebra vanish. The corresponding statement for the hermitian model is that such correlators become infinite or undefined. For $k \to 0$, only the identity and its current algebra descendants have well-defined correlators. (The divergence of correlators of other operators has to do with $k \to 0$, not coincidence of arguments.) We must thus conclude that the wavefunctions can be taken as functions of the current $(\partial H\ H^{-1})_a$.

The above arguments give the conformal field-theoretic reason for the currents being the quantities of interest. The result, however, is not surprising since the Wilson loop operator can be written in terms of H as

$$W(C) \ = \ \mathrm{Tr}\ P\exp\left(-\oint_C dz\ \partial H H^{-1}\right) \tag{20}$$

The Wilson loop operators form a complete set and hence the currents should suffice to generate the gauge-invariant states.

The vacuum state for the kinetic term is given by Ψ_0 =constant. This is normaliz-

able with the inner product (18). From the above arguments, the quantity of interest in constructing higher states is the current

$$\begin{aligned} J_a(x) &= \frac{c_A}{\pi}\left(\partial H\ H^{-1}\right)_a(x) \\ &= \frac{c_A}{\pi}\left[iM^{\dagger}_{ab}(x)A_b(x) + (\partial M^{\dagger}\ M^{\dagger -1})_a(x)\right] \end{aligned} \tag{21}$$

where $M^{\dagger}_{ab} = 2\mathrm{Tr}(t_a M^{\dagger} t_b M^{\dagger -1})$ is the adjoint representation of $M^{\dagger}$. The action of the kinetic energy operator can be calculated as follows.

$$\begin{aligned} T\ \Psi(J) &= -\frac{e^2}{2}\int_z \frac{\delta^2\Psi}{\delta\bar{A}_k(z)\delta A_k(z)} \\ &= -\frac{e^2}{2}\left[\int_{x,y,z}\frac{\delta J_a(x)}{\delta A_k(z)}\frac{\delta J_b(y)}{\delta\bar{A}_k(z)}\frac{\delta^2\Psi}{\delta J_a(x)\delta J_b(y)} + \int_{x,z}\frac{\delta^2 J_a(x)}{\delta\bar{A}_k(z)\delta A_k(z)}\frac{\delta\Psi}{\delta J_a(x)}\right] \end{aligned} \tag{22}$$

From the definition of J_a and $M^{\dagger}$ we get

$$\frac{\delta J_a(x)}{\delta A_k(z)} = i\frac{c_A}{\pi}M^{\dagger}_{ak}(x)\delta(x-z) \tag{23a}$$

$$\frac{\delta J_b(y)}{\delta\bar{A}_k(z)} = -i\frac{c_A}{\pi}\left(\mathcal{D}_y\bar{G}(y,z)M^{\dagger}(z)\right)_{bk} \tag{23b}$$

$$\mathcal{D}_{mn} = \partial\delta_{mn} + i\frac{\pi}{c_A}f_{mnc}J_c \tag{23c}$$

Using (23a) and the definition of $M^{\dagger}$, we find

$$\begin{aligned} \int_z \frac{\delta^2 J_a(x)}{\delta\bar{A}_k(z)\delta A_k(z)} &= i\frac{c_A}{\pi}\left[\frac{\delta M^{\dagger}_{ab}(x)}{\delta\bar{A}_b(y)}\right]_{y\to x} \\ &= \frac{c_A}{\pi}M^{\dagger}_{am}\mathrm{Tr}\left[T^m\bar{D}^{-1}(y,x)\right]_{y\to x} \end{aligned} \tag{24}$$

The coincident limit of $\bar{D}^{-1}$ which appears in this equation has to be evaluated by regularization.The arguments are the same because both functional derivatives act at the same point. We can consider the kinetic energy operator as $E^k_z(z)E^k_{\bar{z}}(z'),\ z'\to z$. This would give a point-splitting regularized version of $\bar{D}^{-1}$. However, it is not gauge-invariant; we need a phase factor connecting the two points z and z'. The covariantly regularized expression can be written as

$$\begin{aligned} T &= 2e^2\int_z\left[E^k_z(z)P(z,z')^{kl}E^l_{\bar{z}}(z')\right]_{z'\to z} \\ P(z,z') &= \exp\left[-A(z-z') - \bar{A}(\bar{z}-\bar{z}')\right] \end{aligned} \tag{25}$$

In $P(z,z')$, the potentials A and $\bar{A}$ are evaluated at $\frac{1}{2}(z+z')$; this midpoint specification is consistent with the hermiticity of T. Notice also that, in this case, $E_z(z)P(z,z') = P(z,z')E_z(z)$. Using (25), we get

$$\int_z \left[\frac{\delta^2 J_a(x)}{\delta \bar{A}_k(z)\delta A_k(z)}\right]_{Reg} = \frac{c_A}{\pi} M^\dagger_{am} \mathrm{Tr}\left[T^m \bar{D}^{-1}(z,x) P^T(z,x)\right]_{z\to x} \tag{26}$$

(P^T is the transpose of P.) We see that the use of expression (25) for T is equivalent to covariant point-splitting regularization of $\bar{D}^{-1}$. Using (17), we finally get

$$\int_z \frac{\delta^2 J_a(x)}{\delta \bar{A}_k(z)\delta A_k(z)} = -\frac{c_A}{\pi} J_a(x) \tag{27}$$

Combining (22,23,27), the kinetic energy operator T is obtained as

$$\begin{aligned} T\ \Psi &= m\left[\int_x J_a(x)\frac{\delta}{\delta J_a(x)} + \int_{x,y} \Omega_{ab}(x,y)\frac{\delta}{\delta J_a(x)}\frac{\delta}{\delta J_b(y)}\right]\Psi \\ \Omega_{ab}(x,y) &= \left[\frac{c_A}{\pi}\delta_{ab}\partial_y \bar{G}(x,y) - i f_{abc} J_c(y)\bar{G}(x,y)\right] \end{aligned} \tag{28}$$

where $m = e^2 c_A/2\pi$. In particular, we see that J_a is an eigenfunction of T with eigenvalue m, i.e.,

$$T\ J_a(x) = \frac{e^2 c_A}{2\pi}\ J_a(x) \tag{29}$$

Of course, J_a by itself would not be an acceptable eigenfunction since it is not invariant under $H \to V(z)H\bar{V}(\bar{z})$. We have to construct suitable combinations of J_a's. Nevertheless, the result (29) is the mathematical expression of the intuitive arguments given earlier. (Expression (28) is also typically of the form which arises in change of variables or the introduction of collective coordinates in field theory [4].)

The ambiguity in defining M for a given potential A is a constant matrix $\bar{V}$ if we use boundary conditions on M appropriate to a Riemann sphere. However, from the point of view of constructing M from A, there is no reason why this should not be done independently in different regions of space with matching conditions on overlap regions. Thus we must allow the freedom of making transformations $H \to V(z)H\bar{V}(\bar{z})$ for the purpose of matching H's in different regions. Proper wave functions are thus constructed from products of the currents by requiring this invariance as well. (This point was not elaborated upon in [1].)

The wavefunction for the simplest excited state is given by the product of two currents. We find, using (28), that

$$\alpha = J_a(x)J_a(y) + \frac{c_A \text{dim}G}{\pi^2}\frac{1}{(x-y)^2} \tag{30}$$

is orthogonal to the ground state and is an eigenfunction of T with eigenvalue $2m$. (dim$G$ is the dimension of the group $G = SU(N)$.) By applying $\bar{\partial}_x$, $\bar{\partial}_y$ and taking $y \to x$, we can construct a state Ψ_2 of eigenvalue $2m$, which is invariant under $H \to V(z)H\bar{V}(\bar{z})$.

$$\Psi_2(J) = \int_x f(x)\left[\bar{\partial}J_a(x)\bar{\partial}J_a(x) + \frac{c_A \text{dim}G}{\pi^2}\partial_x\bar{\partial}_x\delta(x-y)|_{y\to x}\right] \tag{31}$$

The second (c-number) term in (31) orthogonalizes this with respect to the ground state. (One may regard $\bar{\partial}J_a(x)\bar{\partial}J_a(y)$ as providing a point-split version of $B_a(x)B_a(x)$. A point-splitting respecting invariance under $H \to V(z)H\bar{V}(\bar{z})$ would be

$$\beta(x,y) = B_a(x)[P\exp(-\int_y^x \partial HH^{-1})]_{ab}B_b(y) \tag{32}$$

In the limit $y \to x$, $\beta(x,y)$ gives (29) and $T\beta$ goes to $T\Psi_2$ as well.) The construction of higher excited states will be discussed elsewhere.

In ref.[1], an expression for T was given in terms of derivatives with respect to the parameters φ^a of H as defined after (11). The matrix element of the kinetic energy term is given by

$$\langle 1|T|2\rangle = \frac{e^2}{2}\int d\mu(H)e^{2c_A\mathcal{S}(H)}\int_x [(Gp_a\Psi_1)^* K_{ab}(Gp_b\Psi_2)] \tag{33}$$

where

$$\begin{gathered} p_m = -ir^{-1}_{mn}\frac{\delta}{\delta\varphi^n}, \qquad \bar{p}_m = -ir^{*-1}_{mn}\frac{\delta}{\delta\varphi^n} \\ K_{ab} = 2\text{Tr}(t_aHt_bH^{-1}) \end{gathered} \tag{34}$$

The action of T on products of J's can also be computed from this. Taking Ψ_2 to be a function of the current, we get

$$\begin{aligned} \langle 1|T|2\rangle &= i\frac{e^2c_A}{2\pi}\int d\mu(H)e^{2c_A\mathcal{S}(H)}\int_x\left[(Gp_a\Psi_1)^*K_{ab}K_{cb}\frac{\delta\Psi_2}{\delta J_c}\right] \\ &= im\int d\mu(H)\ \left[\Psi_1^*\int_x \bar{G}\bar{p}_a\left\{e^{2c_A\mathcal{S}(H)}\frac{\delta\Psi_2}{\delta J_a}\right\}\right] \\ &= im\int d\mu(H)e^{2c_A\mathcal{S}(H)}\ \left[\Psi_1^*\int_x\left\{-iJ_a(x) + \int_y[(\bar{G}\bar{p}_a)(x)J_b(y)]\frac{\delta}{\delta J_b(y)}\right\}\frac{\delta\Psi_2}{\delta J_a(x)}\right] \end{aligned} \tag{35}$$

We have used the relation $(Gp_b)(x)J_c(y) = (ic_A/\pi)K_{cb}\delta(x-y)$ and $(\bar{G}\bar{p}_a)\mathcal{S} = (-i/2\pi)(\partial HH^{-1})_a$. Using also $(\bar{G}\bar{p}_a)(x)J_b(y) = -i\Omega_{ab}(x,y)$, we see that (35) leads to exactly the same expression as (28). Since it suffices, by our earlier arguments, to consider only wave functions which are functions of the currents, it follows that the operator T as given by (33) is the same as (25) or (28), giving an alternative confirmation of the calculations in [1]. Further, T as given by (33) is evidently self-adjoint; thus (28) is self-adjoint as well, despite the naive lack of manifest hermiticity. In collective field theory, rather than demonstrating self-adjointness, one usually determines the measure factor appearing in the inner product by requiring self-adjointness of the Hamiltonian [4]. It is clear from the above calculations that the measure so determined will lead to the inner product (18).

It is interesting to note that the potential energy can also be written in terms of J_a's as

$$V\ \Psi = \left[\frac{1}{m}\frac{\pi}{c_A}\int_x \bar{\partial}J_a(x)\bar{\partial}J_a(x)\right]\ \Psi \tag{36}$$

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